OCR
RECOGNISING ACHIEVEMENT

OXFORD CAMBRIDGE AND RSA EXAMINATIONS

Advanced Subsidiary General Certificate of Education
Advanced General Certificate of Education

MEI STRUCTURED MATHEMATICS **2610/1**

Differential Equations (Mechanics 4)

Wednesday **22 JUNE 2005** Afternoon 1 hour 20 minutes

Additional materials:
Answer booklet
Graph paper
MEI Examination Formulae and Tables (MF12)

TIME 1 hour 20 minutes

INSTRUCTIONS TO CANDIDATES

- Write your name, centre number and candidate number in the spaces provided on the answer booklet.
- Answer any **three** questions.
- You are permitted to use a graphical calculator in this paper.

INFORMATION FOR CANDIDATES

- The allocation of marks is given in brackets [ ] at the end of each question or part question.
- You are advised that an answer may receive no marks unless you show sufficient detail of the working to indicate that a correct method is being used.
- Final answers should be given to a degree of accuracy appropriate to the context.
- Take $g = 9.8 \text{ m s}^{-2}$ unless otherwise instructed.
- The total number of marks for this paper is 60.

This question paper consists of 4 printed pages.

1 In the simultaneous differential equations

$$\frac{dx}{dt} = -4x + 6y + 28 \qquad (1)$$

$$\frac{dy}{dt} = -3x + 2y + 26 \qquad (2)$$

x and y are the quantities of compounds produced in a chemical reaction.

(i) Eliminate y from the equations to show that $\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 10x = 100$. [4]

(ii) Find the general solution for x. Use this solution and equation (1) to find the corresponding general solution for y. [8]

You are given that $x = y = 0$ when $t = 0$.

(iii) Find the particular solutions for x and y. [3]

(iv) Sketch the solution curves, indicating the long-term values of x and y. Explain how the long-term values could have been found without solving the differential equations. [5]

2 Anna is downloading a file from the internet. She observes that for the file, the download rate, r, is initially 56 kilobytes per second, but reduces during the download. She notices that the rate eventually settles down to 4 kilobytes per second. I is the total amount of data downloaded, in kilobytes, by time t seconds.

(i) Sketch a possible graph of r against t. Sketch on separate axes a possible graph of I against t. [2]

Anna models the rate of decrease of r as proportional to the difference between r and the long-term value of 4 kilobytes per second.

(ii) Show that this model leads to the relation $r = 4(1 + 13e^{-kt})$, where k is a positive constant. [7]

(iii) Find an expression for I in terms of k and t. Given that a 3000 kilobyte file takes 620 seconds to download, show that $10k = 1 - e^{-620k}$ and hence deduce that $k \approx 0.1$. [6]

More detailed analysis of r shows that it fluctuates and is better modelled by the equation

$$\frac{dr}{dt} + 0.1r = 0.4 + 0.2\cos 2t.$$

(iv) Solve this differential equation subject to the same initial conditions. You may assume that there is a particular integral of the form $a + b\cos 2t + c\sin 2t$. [5]

3 The differential equation

$$(x+1)\frac{dy}{dx} + 2\sin y = \cos x \qquad (*)$$

is to be solved for $x \geqslant 0$ subject to the condition that $y = 0$ when $x = 0$. The value of y when $x = 0.2$ is required. Two methods for estimating this value are to be considered.

(a) Euler's method is used. The algorithm is given by $y_{r+1} = y_r + hy'_r$, $x_{r+1} = x_r + h$.

The table shows some of the values calculated using a step length of 0.02.

x	y	y'
0	0	1
0.02	0.02	0.940983
0.04	0.038820	0.886135
0.06	0.056542	0.835072
0.08	0.073244	0.787449
0.10	0.088993	0.742958
0.12	0.103852	0.701320
0.14	0.117878	0.662285
0.16	0.131124	
0.18		
0.20		

(i) Calculate the final entries in the table to estimate y when $x = 0.2$. [5]

(ii) Explain, with the aid of a sketch, why the figures for $\frac{dy}{dx}$ in the table suggest that the estimate calculated in part **(a)(i)** is an overestimate. [3]

(b) For small y, the approximation $\sin y \approx y$ can be used to give the approximate differential equation

$$(x+1)\frac{dy}{dx} + 2y = \cos x. \qquad (**)$$

(i) Solve this differential equation subject to the same initial conditions as before to show that

$$y = \frac{(x+1)\sin x + \cos x - 1}{(x+1)^2}.$$

Use this solution to estimate y when $x = 0.2$. [9]

(ii) Use equation (*) and the fact that $\sin y < y$ for $y > 0$ to show that the value found from solving (**) in part **(b)(i)** is an underestimate. [3]

[Turn over

4 The differential equation $\frac{\mathrm{d}^3y}{\mathrm{d}t^3}+2\frac{\mathrm{d}^2y}{\mathrm{d}t^2}-\frac{\mathrm{d}y}{\mathrm{d}t}-2y=4\mathrm{e}^{-3t}$ is to be solved for $t>0$.

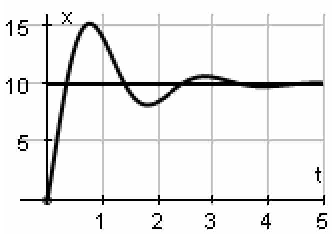
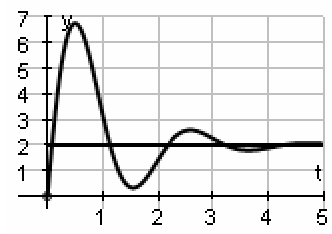
(i) Write down the auxiliary equation. Show that 1 is a root of this equation and find the other two roots. Hence find the general solution of the differential equation. [9]

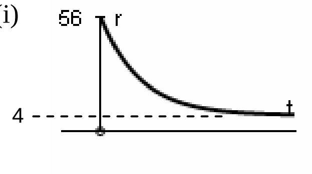
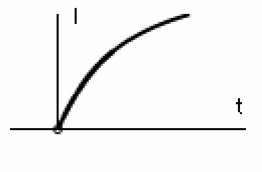
It is given that when $t=0$, $y=-\frac{3}{2}$, $\frac{\mathrm{d}y}{\mathrm{d}t}=\frac{3}{2}$ and also that the solution tends to zero as t tends to infinity.

(ii) Show that the particular solution is $y=-2\mathrm{e}^{-t}+\mathrm{e}^{-2t}-\frac{1}{2}\mathrm{e}^{-3t}$. [5]

(iii) Using the substitution $u=\mathrm{e}^{-t}$, or otherwise, show that the solution found in part **(ii)** is never zero and has no turning points. Hence sketch the solution. [6]

Mark Scheme 2610
June 2005

1(i)	$\frac{d^2x}{dt^2} = -4\frac{dx}{dt} + 6\frac{dy}{dt}$	M1	differentiate	
	$= -4\frac{dx}{dt} + 6(-3x + 2y + 26)$	M1	substitute for $\frac{dy}{dt}$	
	$= -4\frac{dx}{dt} + -18x + \frac{12}{6}\left(\frac{dx}{dt} + 4x - 28\right) + 156$	M1	substitute for y	
	$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 10x = 100$	E1		
				4
(ii)	$\alpha^2 + 2\alpha + 10 = 0$	M1	auxiliary equation	
	$\alpha = -1 \pm 3j$	A1		
	CF $x = e^{-t}(A\cos 3t + B\sin 3t)$	F1	CF for their solutions	
	PI $x = \frac{100}{10} = 10$	B1		
	GS $x = 10 + e^{-t}(A\cos 3t + B\sin 3t)$	F1	their CF + their PI	
	$\frac{dx}{dt} = e^{-t}(-A\cos 3t - B\sin 3t - 3A\sin 3t + 3B\cos 3t)$	M1	differentiate	
	$y = \frac{1}{6}(\dot{x} + 4x - 28)$	M1	rearrange and substitute	
	$y = 2 + \frac{1}{2}e^{-t}\left((A+B)\cos 3t + (B-A)\sin 3t\right)$	A1	cao	
				8
(iii)	$10 + A = 0 \Rightarrow A = -10$	M1	use $t = 0$ with their $x = 0$	
	$2 + \frac{1}{2}(A+B) = 0 \Rightarrow B = 6$	M1	use $t = 0$ with their $y = 0$ (or their $\dot{x} = 28$)	
	$x = 10 + e^{-t}(6\sin 3t - 10\cos 3t)$ $y = 2 + e^{-t}(8\sin 3t - 2\cos 3t)$	A1	both (cao)	
				3
(iv)		B1	initial condition and asymptote for one graph	
		B1	both generally correct (must start at origin)	
		B1	long-term values	
				
	long-term values can be found by setting $\dot{x} = \dot{y} = 0$ in DE's	M1		
	and solving the resulting equations	A1		
				5

2(i)	 	B1 B1	r starts at 56 and decreases, tending to 4 I starts at 0, gradient is positive but decreases	
				2
(ii)	$\frac{\mathrm{d}r}{\mathrm{d}t} = -k(r-4)$	M1 A1		
	$\int \frac{\mathrm{d}r}{r-4} = -k\int \mathrm{d}t$	M1	separate and integrate	
	$\ln\|r-4\| = -kt + c_1$	A1	all correct	
	$r = 4 + Ae^{-kt}$	M1	rearranging	
	$t = 0, r = 56 \Rightarrow A = 52$	M1	use initial condition	
	$r = 4\left(1+13e^{-kt}\right)$	E1		
				7
(iii)	$I = \int r\mathrm{d}t = 4\left(t - \frac{13}{k}e^{-kt}\right) + c_2$	M1	integrate r	
	$t = 0, I = 0 \Rightarrow c_2 = \frac{52}{k}$	M1	use initial condition	
	$I = 4t + \frac{52}{k}\left(1-e^{-kt}\right)$	A1	cao	
	$3000 = 4\times 620 + \frac{52}{k}\left(1-e^{-620k}\right)$	M1	condition on their I	
	$\Rightarrow 10k = 1-e^{-620k}$	E1	must follow correct I	
	unless k very small, $e^{-620k} \approx 0 \Rightarrow 10k \approx 1 \Rightarrow k \approx 0.1$	E1	independent of other marks	
				6
(iv)	$\alpha + 0.1 = 0 \Rightarrow \alpha = -0.1$ so CF $r = Be^{-0.1t}$	B1		
	for given PI $\frac{\mathrm{d}r}{\mathrm{d}t} = -2b\sin 2t + 2c\cos 2t$			
	$0.1a + (0.1c - 2b)\sin 2t + (0.1b + 2c)\cos 2t = 0.4 + 0.2\cos 2t$	M1	differentiate and substitute	
	$0.1a = 0.4$ $0.1c - 2b = 0$ $0.1b + 2c = 0.2$	M1	compare at least two coefficients and solve	
	$a = 4, b = \frac{2}{401}, c = \frac{40}{401}$	A1		
	conditions $\Rightarrow B = 51.995$, so			
	$r = 51.995e^{-0.1t} + 4 + \frac{2}{401}(\cos 2t + 20\sin 2t)$	A1		
				5

3(a)(i)	$\frac{dy}{dx} = \frac{\cos x - 2\sin y}{x+1}$	M1	may be implied	
	0.16 0.625629	M1	use of algorithm	
	0.18 0.143637 0.591150	A1	$y(0.18)$	
	0.20 0.155460	M1	use of algorithm	
		A1	$y(0.20)$	5
(a)(ii)	dy/dx decreases	B1		
		B1	sketch showing curve and step by step solution	
	For each step, gradient used is greater than dy/dx over interval, hence overestimates y.	E1	convincing argument	3
(b)(i)	$\frac{dy}{dx} + \frac{2}{x+1}y = \frac{\cos x}{x+1}$	M1	rearrange	
	$I = \exp\left(\int \frac{2}{x+1}dx\right) = (x+1)^2$	M1 A1	attempt integrating factor	
	$(x+1)^2\frac{dy}{dx} + 2(x+1)y = (x+1)\cos x$	M1	multiply	
	$(x+1)^2 y = \int (x+1)\cos x dx = (x+1)\sin x - \int \sin x dx$	M1	attempt integration by parts	
	$= (x+1)\sin x + \cos x + A$	A1		
	$x = 0, y = 0 \Rightarrow A = -1$	M1		
	$y = \frac{(x+1)\sin x + \cos x - 1}{(x+1)^2}$	E1		
	$x = 0.2 \Rightarrow y = 0.1517$	B1		9
(b)(ii)	Since $\sin y < y$, replacing $\sin y$ by y in $\frac{dy}{dx} = \frac{\cos x - 2\sin y}{x+1}$	M1	consider effect on dy/dx	
	gives an underestimate for dy/dx.	A1		
	Hence (while $y > 0$), the approx. DE will underestimate y.	E1		3

4(i)	$\alpha^3+2\alpha^2-\alpha-2=0$	M1	auxiliary equation	
	$(\alpha-1)(\alpha^2+3\alpha+2)=0$	M1	factorise or demonstrate 1 is a root	
	$\alpha=1,-1,-2$	A1		
	CF $y=A\mathrm{e}^{-t}+B\mathrm{e}^{-2t}+C\mathrm{e}^{t}$	F1	CF for their roots (must have 3 constants)	
	PI $y=a\mathrm{e}^{-3t}$	B1	correct form	
	$\dot{y}=-3a\mathrm{e}^{-3t}, \ddot{y}=9a\mathrm{e}^{-3t}, \dddot{y}=-27a\mathrm{e}^{-3t}$	M1	differentiate and substitute	
	$-27a+18a+3a-2a=4$	M1	compare coefficients	
	$a=-\frac{1}{2}$	A1		
	$y=A\mathrm{e}^{-t}+B\mathrm{e}^{-2t}+C\mathrm{e}^{t}-\frac{1}{2}\mathrm{e}^{-3t}$	F1	their CF + their PI	
				9
(ii)	decays $\Rightarrow C=0$	B1		
	$t=0, y=-\frac{3}{2}\Rightarrow -\frac{3}{2}=A+B-\frac{1}{2}$	M1	condition on y	
	$\dot{y}=-A\mathrm{e}^{-t}-2B\mathrm{e}^{-2t}+\frac{3}{2}\mathrm{e}^{-3t}$	M1	differentiate	
	$t=0, \dot{y}=\frac{3}{2}\Rightarrow \frac{3}{2}=-A-2B+\frac{3}{2}$	M1	condition	
	$A=-2, B=1$ so $y=-2\mathrm{e}^{-t}+\mathrm{e}^{-2t}-\frac{1}{2}\mathrm{e}^{-3t}$	E1		
				5
(iii)	let $u=\mathrm{e}^{-t}$ so $y=-2u+u^2-\frac{1}{2}u^3=-\frac{1}{2}u(u^2-2u+4)$			
	$u=\mathrm{e}^{-t}\neq 0$ and $u^2-2u+4=(u-1)^2+3>0$	M1	consider quadratic (any valid method)	
	hence $y\neq 0$ for all t	E1	if discriminant used, value or working must be shown must indicate u non-zero	
	$\dot{y}=2\mathrm{e}^{-t}-2\mathrm{e}^{-2t}+\frac{3}{2}\mathrm{e}^{-3t}=\frac{3}{2}u\left(u^2-\frac{4}{3}u+\frac{4}{3}\right)$			
	$=\frac{3}{2}u\left(\left(u-\frac{2}{3}\right)^2+\frac{8}{9}\right)>0$	M1	consider quadratic (any valid method)	
	hence no turning points	E1		
		B1 B1	starts at $-\frac{3}{2}$ and asymptote $y = 0$ Shape (increasing)	
				6

2610 - Differential Equations

General Comments

Many candidates showed a good understanding of the techniques required for this unit. The standard of work shown was generally good. Questions one and four were the most popular choices.

The standard of graph sketching was very variable and centres are asked to give candidates the following advice with regard to sketching solution curves. In this unit, sketches are expected to show the basic features of the solution (e.g. oscillating, increasing, decreasing, decaying, growing, asymptotes), and detailed calculations are not required, unless the question specifically asks for them. However, sketches are expected to show the initial or boundary conditions given in the question and any results found in the course of answering the question.

Comments on Individual Questions

1) (i) This was usually correctly answered, although some candidates differentiated with respect to x rather than t.

(ii) Most candidates knew how to solve the differential equation for x, although some candidates omitted the particular integral. Many candidates used a linear or quadratic form for the particular integral. Although this usually led to a correct answer, it was inefficient. When the right hand side of an equation such as this is constant, candidates are expected to use a constant for the particular integral. Indeed the particular integral can be simply stated as the ratio of the right hand side over the coefficient of x.

(iii) Candidates generally were able to find the particular solutions.

(iv) The sketches were very variable. Often candidates' sketches did not oscillate. Another common error was for the sketches not to start at the origin, despite the given initial conditions. Some candidates omitted the final request to explain how the long-term values could be found without solving the equations.

2) (i) The sketches were often adequate, although some candidates did not show the initial values on their sketches.

(ii) This was often well done. However some candidates were muddled in the sign of their constant of proportionality. It was surprising that some candidates tried to find the solution with no attempt to set up or solve a differential equation.

(iii) This was often not done well. Some candidates did not realise that r needed to be integrated, and those who did often omitted the constant or did not see how to calculate it from the initial conditions.

(iv) Many candidates knew the method required, but accuracy was usually a problem here. Despite the direction given in the question, a sizeable minority of candidates tried to use the integrating factor method, resulting in an integral which few candidates were able to find.

3) (a)(i) The calculations were often done well, but some candidates produced a string of wrong numbers with no evidence of method.

(ii) Most candidates showed some understanding of why the calculation gave an overestimate, but some sketches were unclear, and some did not refer to the significance of the decreasing values of y'.

(b)(i) The solution was often done well, but some candidates omitted the constant of integration, or assumed its value with no method shown.

(ii) This part was rarely done well. Candidates usually did not consider the effect on dy/dx, but concentrated on the effect on y, ignoring the derivative.

4) (i) This was usually done very well, but some candidates only used two of the roots in their complementary function.

(ii) Candidates generally were able to produce two equations using the initial conditions, but some were unable to make use of the condition on the behaviour as t tends to infinity.

(iii) Some candidates did this very well. However some claimed that the expressions for y and dy/dt in terms of u were not zero with little justification. The sketches were generally good, but some omitted the initial value of y.